

The Landau-Lifshitz equation, the NLS, and the magnetic rogue wave as a by-product of two colliding regular “positons”

A. V. Yurov^{*†} V. A. Yurov^{*‡}

Abstract

In this article we present a new method for construction of exact solutions of the Landau-Lifshitz-Gilbert equation (LLG) for ferromagnetic nanowires. The method is based on the established relationship between the LLG and the nonlinear Schrödinger equation (NLS), and is aimed at resolving an old problem: how to produce multiple-rogue wave solutions of NLS using just the Darboux-type transformations. The solutions of this type - known as P-breathers – have been proven to exist by Dubard and Matveev, but their technique heavily relied on using the solutions of yet another nonlinear equation, Kadomtsev-Petviashvili I equation (KP-I), and its relationship with NLS. We have shown that in fact one doesn’t have to use KP-I but can instead reach the same results just with NLS solutions, but only if they are dressed via the *binary* Darboux transformation. In particular, our approach allows to construct all the Dubard-Matveev P-breathers. Furthermore, the new method can lead to some completely new, previously unknown solutions. One particular solution that we have constructed describes two “positon”-like waves, colliding with each other and in the process producing a new, short-lived rogue wave. We called this unusual solution (rogue wave begotten after the impact of two solitons) the “impacton”

1 Introduction

One of the primary tools essential for studying the magnetic properties of ferromagnetic solids is the famous Landau-Lifshitz-Gilbert (LLG) equation [1], [2]. Its purpose lies in describing the dynamics of a magnetization vector field \vec{M} of the ferromagnetic in response to an effective magnetic field \vec{H}_{eff} (which includes an external magnetic field, and the effects of anisotropy and of spin exchange interaction) and an electric current j_e ¹ and have a form (t being the time variable):

$$\frac{\partial \vec{M}}{\partial t} = -\gamma \vec{M} \times \vec{H}_{\text{eff}} + \frac{\alpha}{M_s} \vec{M} \times \frac{\partial \vec{M}}{\partial t} + \vec{F}_j, \quad (1)$$

^{*}Immanuel Kant Baltic Federal University, Department of Physics and Technology, Al.Nevsky St. 14, Kaliningrad, 236041, Russia

[†]AIUrov@kantiana.ru

[‡]vayt37@gmail.com

¹It is also possible to account for some additional effects such as a presence of magnetic defects right in (1) by simply adding the required terms to the effective field \vec{H}_{eff} (see, for example, [3]).

where γ is the gyromagnetic ratio, α is called the Gilbert dampening parameter, M_s is the saturation magnetization and \vec{F}_j is called the spin-transfer torque. The equation describes the dynamics of the field \vec{M} as a precession around the field \vec{H}_{eff} (the first term), combined with gradual dissipation of all components of \vec{M} orthogonal to \vec{H}_{eff} (the second term, which acts on \vec{M} as a sort of a “viscous force”). The last term accounts for the effect of polarization P of an electric current j that goes through the ferromagnetic, and for the torque this polarization inflicts on the overall magnetization.

In order to solve (1) one has to first define the geometry of the problem as well as the relative strength of parameters γ and α , followed by subsequent rewriting of the vector equation as a system of scalar differential equations. Most astonishingly, many of those geometries lead to one very specific and famous nonlinear equation: the *nonlinear Schrödinger equation* (NLS). This is not a coincidence, as the equivalence between the LLG and NLS has already been demonstrated as early as in 1970s [4], [5], and this connection has been extended for some generalized NLS models as well [6]. Just as a few particular examples, the LLG written for ferromagnetic nanowires (with no damping) reduces to straightforward NLS [7]; magnetization of a one-dimensional multicomponent magnonic crystal is shown to be governed by NLS with spatially-dependent coefficients [8]; finally, the flat multilayered ferromagnetic crystal whose easy axis is normal to the layers can also be reduced to a standard NLS with a potential containing a sequence of δ -functions [9].

In this article we will study a simpler of these cases: the ferromagnetic nanowire. For simplicity, we will assume that the nanowire is parallel to x -axis, and that its anisotropic term and external field H_{ext} has only z components (i.e. that we are working with a perpendicular anisotropic ferromagnetic nanowire), just like the one described in [7]. In this case the spin exchange interaction will produce the term proportional to \vec{M}_{xx} , the anisotropy (with good accuracy) will be proportional to M_z (z -component of \vec{M}), and the current along the nanowire will produce the spin-transfer torque proportional to \vec{M}_x :

$$\frac{\partial \vec{M}}{\partial t} = -\gamma \vec{M} \times \left(\frac{2K}{M_s^2} \frac{\partial^2 \vec{M}}{\partial x^2} + \left(\frac{K}{M_s} - 4\pi \right) M_z \vec{k} + H_{\text{ext}} \vec{k} \right) + \frac{\alpha}{M_s} \vec{M} \times \frac{\partial \vec{M}}{\partial t} + \frac{Pj\mu_B}{eM_s} \frac{\partial \vec{M}}{\partial x}, \quad (2)$$

where \vec{k} is a z -axis unit vector, B is the exchange constant, K is the anisotropy coefficient, μ_B is the Bohr magneton and e is the electron charge. The first thing we notice here is that the equation (2) can be simplified by a simple rescaling $\vec{M} \rightarrow \vec{m}$, $t \rightarrow \tau$ and $x \rightarrow \xi$:

$$\vec{m} = \frac{\vec{M}}{M_s}, \quad \tau = \frac{t}{t_0} = t\gamma(K - 4\pi M_s), \quad \xi = \frac{x}{x_0} = x \sqrt{\frac{K - 4\pi M_s}{2B}}, \quad (3)$$

which produces the following dimensionless equation

$$\frac{\partial \vec{m}}{\partial \tau} = -\vec{m} \times \left(\frac{\partial^2 \vec{m}}{\partial \xi^2} + (m_z + h_{\text{ext}}) \vec{k} \right) + \alpha \vec{m} \times \frac{\partial \vec{m}}{\partial \tau} + B_j \frac{\partial \vec{m}}{\partial \xi}, \quad (4)$$

where we have introduced the following new notation:

$$h_{\text{ext}} = \frac{H_{\text{ext}}}{K - 4\pi M_s}, \quad B_j = \frac{Pj\mu_B}{eM_s\gamma\sqrt{2B(K - 4\pi M_s)}} = \frac{Pj\mu_B}{eM_s} \frac{t_0}{x_0}.$$

Next, recalling that $|\vec{m}| \leq 1$, let us introduce a new complex-value function $q(\xi, \tau)$ that is related to the normalized magnetization field as follows:

$$m_x + im_y = q, \quad m_z^2 = 1 - |q|^2. \quad (5)$$

Finally, let us make a reasonable two reasonable assumption regarding the magnitude of $|q|^2$: let's assume that z component of magnetization is much stronger than the other two, so $|q|^2 \ll 1$ (the so-called long-wave approximation, [10]). This condition implies that we can keep only those nonlinear terms in (4) that are of the magnitude $|q|^2 q$. Furthermore, if the dampening factor is also very small ($\alpha \ll 1$ and does not exceed the maximal value of q), we would also be able to omit the dampening term altogether, thus ending up with the following equation:

$$iq_\tau = q_\xi \xi + \frac{1}{2}|q|^2 q + iB_J q_\xi - \omega_0 q, \quad (6)$$

where $\omega_0 = 1 + h_{\text{ext}}$. The equation (6) is nothing else but a nonlinear Schrödinger equation. The last purely cosmetic changes that we can do here is reduce (6) to a canonic form by introducing the new variables $\tau \rightarrow t'$, $\xi \rightarrow x'$:

$$\tau = 4t', \quad \xi = 2x' - 4B_J t', \quad q = e^{i\omega_0 \tau} \bar{u}(t', x'), \quad (7)$$

which allows one to obtain for $u(t, x)$ the standard NLS equation (10):

$$iu_{t'} + u_{x'x'} + 2|u|^2 u = 0, \quad (8)$$

where for all further calculations we will omit the accents $'$ and will write just x and t .

The equation (8) has been under a careful scrutiny ever since its inception, so as a result we now know quite a lot about its possible solutions (“bright” and “dark” solitons, breathers, N-soliton solutions etc.). However, perhaps the most interesting solutions are the so-called rogue wave solutions. First discovered in 1983 by Peregrine [11] and being the first known completely localized regular solution of a form:

$$u(x, t) = \left(1 - \frac{4(1 + 4it)}{1 + 4x^2 + (4t)^2} \right) e^{it}. \quad (9)$$

This solution, called the Peregrine soliton, asymptotically behaves as a simple plane wave, but it also have the exact behaviour of the dreaded oceanic “rogue waves”, example of which has been detected in the North Sea on the very first day of 1995, inflicting some (thankfully) minor damage to the Draupner platform (the offshore hub for Norway’s gas pipelines) [12]. In later years, the “rouge waves” have become a subject of active explorations and experiments; they have been observed in the optical fibers [13], in the waves generated in the multicomponent plasma [14], and even in the experimental water tank [15]. So, it was only a matter of time until the attention would turn to the possibility of rogue waves forming more complex structures, possibly consisting of multiple rogue waves. However, until recently all the attempts to construct such solutions have been nothing but exercises in futility, as it soon became clear that the standard techniques – such as the Darboux transformation (see Sections 2 and 4 for more information) – fail to produce any Peregrine soliton-like solutions past the already known ones (the exact reasons for this will be discussed in Sections 3 and 4). This vicious cul-de-sac has only been broken in [17], published in 2010, which proposed a way to actually construct the multi-rogue waves solution by essentially going around the obstacles and working not with NLS, but with another equation, Kadomtsev-Petviashvili I (KP-I), and afterwards using the relationship between NLS and KP-I to get the required NLS solutions. This technique has allowed the authors to construct a set of new solutions, including so-called “P”-breathers.

However, in this article we will adopt a different approach by sticking to NLS and demonstrating that it is still possible to achieve the same goal using the techniques of

Darboux transformation – provided one uses not a standard, but a *binary* Darboux transformation. Furthermore, we will show that our approach not only allows to get the same P-breathers as in [17], [18] but also a previously unknown solution with a very unique properties: it describes a collision of two slowly-moving regular positons (or negatons), which produces a short-lived rogue wave, which we have tentatively called the “*impacton*”. One interesting aspect of the impacton model lies in slowness of movement of its parental solitons which might make this solution a very good candidate for observation on particular ferromagnetic nanowires.

2 NLS and its zero-curvature condition

In this section we begin the exploration of the NLS equation

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad (10)$$

and will attempt to answer the question posed in the introduction: does there exist a simple way to produce a new non-trivial Peregrine-style solution from the already known one?

In order to succeed in our endeavour we will first have to reduce our nonlinear problem to a slightly more manageable linear system. The system in question is the zero curvature condition, sometimes also called the Lax pair for the NLS.

$$\begin{aligned} \Psi_x &= i\sigma_3\Psi\Lambda + iU\Psi, \\ \Psi_t &= 2i\sigma_3\Psi\Lambda^2 + 2iU\Psi\Lambda + W\Psi, \end{aligned} \quad (11)$$

where Λ is a 2×2 diagonal complex-valued matrix, σ_3 is Hermitian, and is called the third Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (12)$$

$\Psi = \Psi(x, t)$ is a 2×2 matrix-valued function, and the matrices U , V are defined as follows:

$$U = \begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix} \quad W = \begin{pmatrix} -i|u|^2 & \bar{u}_x \\ -u_x & i|u|^2 \end{pmatrix} = \sigma_3(U_x - iU^2). \quad (13)$$

It is important to note, that the zero-curvature condition for (10) can also be rewritten as a conjugate system for a *different* 2×2 matrix-valued function $\Phi = \Phi(x, t)$ and different spectral parameter μ (independent of λ):

$$\begin{aligned} \Phi_x &= iM\Phi\sigma_3 - i\Phi U, \\ \Phi_t &= -2iM^2\Phi\sigma_3 + 2iM\Phi U - \Phi W. \end{aligned} \quad (14)$$

Furthermore, it is easy to see that the matrix U can be rewritten as

$$U = \text{Re}(u) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \text{Im}(u) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \text{Re}(u)\sigma_1 + \text{Im}(u)\sigma_2,$$

where σ_1 and σ_2 are the first and second Pauli matrices, and just like σ_3 , they are Hermitian and unitary. Thus, U is itself Hermitian. Similarly, it is easy to show that the matrix W is skew-Hermitian, i.e.:

$$U^+ = U, \quad W^+ = -W, \quad (15)$$

where $^+$ indicates the conjugate transpose. (15) implies that for known Ψ and Λ the choice

$$\Phi = \Psi^+, \quad M = -\Lambda^+, \quad (16)$$

will automatically satisfy the conjugate system (14).

Next, we will need an additional tool, designed with the Darboux transformation in mind: a closed 1-form $\Omega = \Omega(\Phi, \Psi)$, that satisfies both the condition

$$\Phi\Psi = i(M\Omega + \Omega\Lambda), \quad (17)$$

and the differential equation

$$\Omega_x = \Phi\sigma_3\Psi. \quad (18)$$

In order to close the 1-form, that is for it to satisfy the compatibility condition

$$\Omega_{tx} = \Omega_{xt}, \quad (19)$$

the function Ω should satisfy the following easily verifiable condition:

$$\Omega_t = 2i(\Phi_x\Psi - \Phi\Psi_x) - 2\Phi U\Psi = 2(\Phi\sigma_3\Psi\Lambda - M\Phi\sigma_3\Psi) + 2\Phi U\Psi. \quad (20)$$

Moving ever closer to the task of construction of the Darboux transformation, let us assume that in addition to Ψ and Φ , we also have two other functions Ψ_1 and Φ_1 , that are also solutions to (11), (14) albeit for the different spectral matrices Λ_1 and M_1 correspondingly. These functions produce two supplementary matrices τ and σ :

$$\tau = \Psi_1\Lambda_1\Psi_1^{-1}, \quad \sigma = \Phi_1^{-1}M_1\Phi_1, \quad (21)$$

that are notable for satisfying the following conditions (with brackets denoting the commutator):

$$\begin{aligned} \tau_x &= i[\sigma_3, \tau]\tau + i[U, \tau], \\ \tau_t &= 2i[\sigma_3, \tau]\tau^2 + 2i[U, \tau]\tau + [W, \tau], \end{aligned} \quad (22)$$

and

$$\begin{aligned} \sigma_x &= i\sigma[\sigma, \sigma_3] + i[U, \sigma], \\ \sigma_t &= 2i\sigma^2[\sigma_3, \sigma] - 2i\sigma[U, \sigma] + [W, \sigma]. \end{aligned} \quad (23)$$

With all these pieces now in place we are finally free to define a new Darboux transformation, that would utilize both the supplementary one-form Ω and the support function Ψ_1 :

$$\begin{aligned} \Phi &\rightarrow \Phi^{(+1)} = \Omega(\Phi, \Psi_1)\Psi_1^{-1} \\ \Phi_1 &\rightarrow \Phi^{(+1)} = \Omega(\Phi_1, \Psi_1)\Psi_1^{-1}. \end{aligned} \quad (24)$$

Such transformation will naturally affect all the remaining ingredients of (11) and (14), transforming them via the following easily verifiable formulas:

$$\begin{aligned} \Psi &\rightarrow \Psi^{(+1)} = \Psi\Lambda - \tau\Psi \\ U &\rightarrow U^{(+1)} = U + [\sigma_3, \tau] = U + 2\sigma_3\tau \\ W &\rightarrow W^{(+1)} = W + 2i(U^{(+1)}\tau - \tau U) = \sigma_3(U_x^{(+1)} - i(U^{(+1)})^2). \end{aligned} \quad (25)$$

We would like to point out at this step that the Darboux transformation (24), (25) all rely on the support function Ψ_1 , which was a particular solution of (11) with the spectral matrix $\Lambda = \Lambda_1$. This, however, is but one of the possibilities. The alternative way to define our transformation would be via the function Φ_1 from (14), thus producing the following system:

$$\begin{aligned}\Psi &\rightarrow \Psi^{(-1)} = \Phi_1^{-1} \Omega(\Phi_1, \Psi) \\ \Psi_1 &\rightarrow \Psi_1^{(-1)} = \Phi_1^{-1} \Omega(\Phi_1, \Psi_1) \\ \Phi &\rightarrow \Phi^{(-1)} = M\Phi - \Phi\sigma \\ U &\rightarrow U^{(-1)} = U + [\sigma_3, \sigma].\end{aligned}\tag{26}$$

It is important to point out here that in order for the formulas for $\Phi^{(+1)}$ and $\Phi_1^{(+1)}$ from (24), as well as $\Psi^{(-1)}$ and $\Psi_1^{(-1)}$ from (26) to be satisfied, the condition (17) should hold. In particular, for $\Phi^{(+1)}$ to be true, one should have

$$\Phi\Psi_1 = i(M\Omega(\Phi, \Psi_1) + \Omega(\Phi, \Psi_1)\Lambda_1),\tag{27}$$

whereas the prerequisite for the Darboux transformation $\Psi_1 \rightarrow \Psi_1^{(-1)}$ is

$$\Phi_1\Psi_1 = i(M_1\Omega(\Phi_1, \Psi_1) + \Omega(\Phi_1, \Psi_1)\Lambda_1).\tag{28}$$

Suppose the (28) indeed holds. Then it is possible to utilize both positive and negative Darboux transforms and introduce a new *binary Darboux transformation* (binary DT), which can be defined in two ways:

$$\begin{aligned}U &\rightarrow U^{(+1)} \rightarrow U^{(+1,-1)} \\ U &\rightarrow U^{(-1)} \rightarrow U^{(-1,+1)},\end{aligned}\tag{29}$$

where the first of binary DT is defined as:

$$\begin{aligned}U^{(+1,-1)} &= U^{(+1)} + [\sigma_3, \sigma^{(+1)}], \\ \sigma^{(+1)} &= \left(\Phi_1^{(+1)}\right)^{-1} M_1 \Phi_1^{(+1)} = \Psi_1 \Omega^{-1}(\Phi_1, \Psi_1) M_1 \Omega(\Phi_1, \Psi_1) \Psi_1^{-1},\end{aligned}\tag{30}$$

which can alternatively be rewritten as

$$U^{(+1,-1)} = U + [\sigma_3, \tau + \sigma^{(+1)}] = U - i[\sigma_3, G_{11}],\tag{31}$$

where for simplicity we have introduced a new matrix-valued function G_{11} defined as

$$G_{11} = \Psi_1 \Omega^{-1}(\Phi_1, \Psi_1) \Phi_1.\tag{32}$$

Most astonishingly, by repeating the exact same calculations for the second binary DT $U^{(-1,+1)}$ will produce exactly the same result, thus producing the following extremely important result:

$$U^{(+1,-1)} = U^{(-1,+1)}.\tag{33}$$

This means that the +1 and -1 DT's actually *commute* with each other and their order is inessential. We can therefore introduce the following notation:

$$U \rightarrow U^{(+n,-m)}, \quad n, m \in \mathbb{N},\tag{34}$$

where n and m denote the amount of “positive” and “negative” Darboux transformations applied to U .²

²Naturally, this notation allows to define U itself as $U^{(+0,-0)}$ and the individual “positive and “negative” DT's as $U^{(+1,-0)}$ and $U^{(+0,-1)}$ correspondingly.

3 The Reduction Restriction (and a First Snag)

In the previous chapter we have demonstrated the existence of not just one but *two* Darboux transformations: $U \rightarrow U^{(+1)}$ and $U \rightarrow U^{(-1)}$ and have shown that together they form a binary DT $U \rightarrow U^{(+1,-1)}$, and, indeed, a whole slew of iterative binary DT's that we have denoted as $U^{(+n,-m)}$. However, the corresponding DT's all share a similar problem: in general they do not respect the Hermitian condition (15). In this chapter we would like to amend this little snag by restricting our attention to just those binary DT's that produce the Hermitian matrices, that is, satisfy the condition

$$(U^{(+n,-m)})^+ = U^{(+n,-m)}, \quad \text{for } \forall n, m \in \mathbb{N}, \quad (35)$$

which we will henceforth call the *reduction restriction* condition.

We will begin by examining the DT we introduced first: the $U^{(+1)}$. To be specific, let us define the individual elements of matrices Ψ_1 and Λ_1 as:

$$\Psi_1 = \begin{pmatrix} \psi_1 & \psi_2 \\ \phi_1 & \phi_2 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}. \quad (36)$$

Fortunately, we don't have to work with all four individual elements of matrix Ψ_1 , since they exist the following famous (and easily verifiable) reduction [16]:

$$\psi_1 = \bar{\phi}_2, \quad \psi_2 = -\bar{\phi}_1. \quad (37)$$

Moreover, we can significantly simplify our calculations by choosing as simple seed solution $u(t, x)$ of (10) as possible. In our case we will work with the periodic solution

$$u = Ae^{iS}, \quad S = ax + (2A^2 - a^2)t, \quad a, A \in \mathbb{R}. \quad (38)$$

From (36), (37) and (38) substituted into (11) we immediately obtain the following partial differential equations on ϕ_1 and ϕ_2 :

$$\begin{aligned} \phi_{1x} &= -i\mu\phi_1 + iu\bar{\phi}_2 \\ \phi_{2x} &= -i\bar{\mu}\phi_2 - iu\bar{\phi}_1 \\ \phi_{1t} &= i(A^2 - 2\mu^2)\phi_1 - iu(2\mu - a)\bar{\phi}_2 \\ \phi_{2t} &= i(A^2 - 2\bar{\mu}^2)\phi_2 + iu(a - 2\bar{\mu})\bar{\phi}_1. \end{aligned} \quad (39)$$

Applying the DT (25) will produce a new NLS solution $u^{(1)}$ that will have a form

$$u^{(1)} = u + \frac{2(\bar{\mu} - \mu)\phi_1\phi_2}{|\phi_1|^2 + |\phi|^2}, \quad (40)$$

and a similar formula for $\bar{u}^{(1)}$, indeed producing the matrix $U^{(+1)}$ in a Hermitian form, and therefore substantiating the previous reduction (37). The particular solutions for ϕ_1 and ϕ_2 for the resulting problem will be of the form [16]

$$\phi_i = f_i e^{iS/2}, \quad i = 1, 2, \quad (41)$$

where f_i are real-valued and S is the same as in (38). After the substitution of (41) into the linear system (39) we will have a number of possibilities open; one that we are

particularly interested in corresponds to the case when the roots of the characteristic equations on f_i are both equal to zero, which happens when

$$\mu = -\frac{a}{2} \pm iA. \quad (42)$$

Not surprisingly, the resulting functions f_i will be linear in terms of both x and t variables:

$$f_1 = bx + 2b(iA - a)t + c, \quad f_2 = 2\bar{b}(ia - A)t - i\bar{b}x + i\left(\frac{\bar{b}}{A} - \bar{c}\right), \quad (43)$$

What *is* interesting, is that upon the Darboux transformation (40) and the subsequent simplifications, we will end up with a *localized* solution: the Peregrine breather, which is essentially a Peregrine soliton on a the background of a planar wave (38),

$$u^{(1)} = Ae^{iS} \left(-1 + \frac{2(1 + 4itA^2)}{2A^2\eta^2 - 2A\eta + 8A^4t^2 + 1} \right), \quad (44)$$

$$|u^{(1)}|^2 = A^2 + \frac{8A^3(4A^3t^2 + \eta - A\eta^2)}{(2A^2\eta^2 - 2A\eta + 8A^4t^2 + 1)^2},$$

with $\eta = x - 2at$, $b = 1$, $c = 0$ and we choose the down sign ("−") in (42). Thus, we actually end with a very simple and straightforward mechanism for “construction” of the Peregrine solitons for NLS. Reverting the steps taken in Section (1) and returning from u back to magnetization \vec{M} , will therefore yield us what the authors of [7] has called the magnetic rogue wave (however, as we can now see, theirs was basically just a rediscovery of a well-known result of Matveev and Salle [16], merely redressed for the LLG).

Unfortunately, this simple algorithm is literally a single-shot weapon! A stumbling block here is the condition (42). Although it allows us to produce a Peregrine breather after one DT, for our purposes it is not enough. In order to build a multisoliton Pererine-like solution, we should use multiple iterations of DT, and so we must have at least *two* linearly independent Peregrine solitons. But each one of them would have to be constructed with a spectral restriction (42) in mind; and it can be shown that, regardless of the sign in (42), the resulting Peregrine solutions will always be linearly dependent, thus producing nothing but zero after the second iteration of DT.

One way around this obstacle has been proposed by Dubard and Matveev in [18] where they have used the relationship existing between the focusing NLS and the Kadomtsev-Petviashvili I (KP-I) equation to produce a multi-rogue waves solutions for NLS (and, subsequently, a family of localized rational solutions of KP-I). Hover, in this article we will adopt a different strategy and demonstrate that it is actually possible to develop a strategy for construction of a multi-rogue wave profile NLS solution while remaining entirely in the framework of Darboux transformation for NLS. The key here is to use not juts DT, but a *binary DT*!

4 Understanding the Binary DT: from the Stationary Schrödinger Equation to KdV

Before we proceed it would be beneficial to take a glance at a theory of binary DT for a simpler equation than NLS: the stationary Schrödinger equation. Suppose ψ and ϕ are

some linearly independent solutions for the Schrödinger equation with the same potential $v = v(x)$, albeit with different spectral parameters λ and μ :

$$\begin{aligned}\psi_{xx} &= (v - \lambda)\psi \\ \phi_{xx} &= (v - \mu)\phi.\end{aligned}\tag{45}$$

Then out of these two solutions one can construct a new solution $\psi^{(1)}$ to the Schrödinger equation

$$\psi_{xx}^{(1)} = (v^{(1)} - \lambda)\psi^{(1)},$$

where the new solution and a new potential $v^{(1)}$ are calculated via the Darboux transformation:

$$\psi \rightarrow \psi^{(1)} = \psi_x - \frac{\phi_x}{\phi}\psi, \quad v \rightarrow v^{(1)} = v - 2(\ln \phi)_{xx}.\tag{46}$$

It is quite apparent from (46) that the linear independence of ϕ and ψ is a crucial condition, as its violation produces only a trivial solution $\psi^{(1)} \equiv 0$. This condition is automatically satisfied when we choose $\lambda \neq \mu$. But what if they are identical, as was the case with the Peregrine solitons in the previous chapter?.. Is DT useless in this case?.. Actually, the answer is *no*, and it has to do with the fact that the Schrödinger equation is a second order linear O.D.E., and therefore must necessary have two linearly independent solutions for any value of a spectral parameter.³ Suppose, for example, that we are looking for a solution $\tilde{\phi}$ that is linearly independent of ϕ but satisfies the same equation with the same spectral parameter. Using the well-known identity

$$\frac{d}{dx} \left(\frac{\tilde{\phi}}{\phi} \right) = \frac{\Delta(\tilde{\phi}, \phi)}{\phi^2},$$

where Δ is a Wronskian of the solutions $\tilde{\phi}$ and ϕ , and the fact that for the Schrödinger equation the Wronskian of two solutions is constant⁴ immediately means that

$$\tilde{\phi} = \phi \int \frac{dx}{\phi^2}.\tag{47}$$

This new solutions has a distinction being linearly independent of the original ϕ , and the direct application of the Darboux transformation (46) with ϕ and $\tilde{\phi}$ produces the function that we will call $\tilde{\phi}^{(1)}$:

$$\tilde{\phi}^{(1)} = \tilde{\phi}_x - \frac{\phi_x}{\phi}\tilde{\phi} = \frac{1}{\phi}.\tag{48}$$

As we have discussed above, this will be a solution to a Schrödinger equation with a new potential $u^{(1)}$. This new equation will also have a *second* solution, linearly independent of $\tilde{\phi}^{(1)}$, that can be calculated using (47). This new solution that we will call $\phi^{(1)}$ will have a form:

$$\phi^{(1)} = \frac{1}{\phi} \int \phi^2 dx.\tag{49}$$

³Although only one of them can belong to L_2 space.

⁴This is the result of a following easily verifiable fact: for any homogeneous linear O.D.E.

$$y'' + p(x)y' + q(x)y = 0,$$

the Wronskian of two solutions y_1 and y_2 will always satisfy the condition $\ln \Delta(y_1, y_2) = \int p(x)dx$.

Thus, we end up with two linearly independent solutions of a new Schrödinger equation with a new potential $v^{(1)}$. But then we can use them in the Darboux transformation once again, and produce another solution for yet another new potential $v^{(2)}$:

$$v^{(2)} = v^{(1)} - 2 \left(\ln \phi^{(1)} \right)_{xx} = v - 2 \frac{d^2}{dx^2} \ln \left(\int \phi^2 dx \right), \quad (50)$$

and it is this transformation that is called the *binary Darboux transformation*.

It is interesting to note that (50) allows one to construct positons and negatons solutions of the KdV equation directly. Such solutions were obtained in a set of articles (see for example [19], [20]) via some generalization of DT, namely via the Darboux transformations complemented by the differentiation with respect to a spectral parameter. Importantly, it was the similar approach used in [18] that eventually led to construction of new multi-rogue waves. On the other hand, as we shall soon see, both positon (negaton) solutions of the KdV equations and the multi-rogue waves solutions of the focusing NLS equations might all be obtained just by the binary DT without any additional differentiation. In the remainder of this section, we'll concentrate on showing this for the KdV, so that when we return back to the discussion of NLS equations in next sections we will already have a sort of a point of reference.

The KdV equation ($v = v(x, t)$) has the well known form:

$$v_t - 6vv_x + v_{xxx} = 0,$$

and may be obtained from its Lax pair, which contains one of the equation from the system (45) (let it be the equation for the $\phi = \phi(x, t)$) and an additional evolutionary equation:

$$\phi_t = 2(v + 2\mu)\phi_x - v_x\phi. \quad (51)$$

If we put $v = 0$, $\mu = -\kappa^2$ and $\phi = \cosh \eta$ with $\eta = \kappa(x - 4\kappa^2 t)$ then (46) results in the famous one-soliton solution $v^{(1)} = -2\kappa^2 \operatorname{sech}^2 \eta$. The dressed solution of Lax pair with the same value of the spectral parameter may be calculated with the help of the (49) with one caveat – one has to correctly define the limits of integration in order to be sure that $\phi^{(1)}(x, t)$ is indeed a solution of (51) with $v \rightarrow v^{(1)}$. We choose the upper limit as x and a lower limit as $\omega(t)$ where the function $\omega(t)$ must be obtained. So:

$$\phi^{(1)} = \frac{1}{\phi} \int_{\omega(t)}^x \phi^2 dx = \frac{1}{\kappa \cosh \eta} \int_{\eta_1(t)}^{\eta} \cosh^2 \eta' d\eta' = \frac{2\eta + \sinh 2\eta - 2\eta_1(t) - \sinh 2\eta_1(t)}{4\kappa \cosh \eta}, \quad (52)$$

with $\eta_1(t) = \kappa(\omega(t) - 4\kappa^2 t)$. Substituting (52) into the (51) one gets the simple first order differential equation for the unknown $\eta_1(t)$ which may be integrated, so (the integration constant being omitted for the sake of brevity)

$$2\eta_1(t) + \sinh 2\eta_1(t) = 16\kappa^3 t. \quad (53)$$

Substituting (53) into the (52) and then using (50) (with the correct limits of integration) we get exactly one negaton solution from the [20] (we again omit the constants of integration):

$$v^{(2)} = -2 \frac{\partial^2}{\partial x^2} \ln (\sinh 2\eta + 2\kappa\beta), \quad (54)$$

with $\beta = x - 12\kappa^2 t$. To obtain the positon solution one should simply repeat the described procedure for a positive value of μ .

5 Binary DT and NLS

Let's say we are planning to extend the ideology discussed in the previous chapter to the task of construction of a binary DT for NLS. We will begin by imposing the already familiar restrictions (recall that $^+$ denotes the conjugate transpose):

$$\Phi = \Psi^+, \quad M = -\Lambda^+,$$

and observing that the closed form Ω should then satisfy the following condition:

$$\Psi^+ \Psi = i(\Omega \Lambda - \Lambda^+ \Omega), \quad (55)$$

where, just as before, we will choose

$$\Psi = \begin{pmatrix} \bar{\phi}_2 & -\bar{\phi}_1 \\ \phi_1 & \phi_2 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}, \quad (56)$$

and we denote the entries of the matrix Ω as:

$$\Omega = \Psi_1 = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}. \quad (57)$$

Taken together, (57), (56) and the condition (5) immediately provides the diagonal values of Ω :

$$\Omega_{11} = -\Omega_{22} = \frac{i}{\bar{\mu} - \mu} (|\phi_1|^2 + |\phi_2|^2). \quad (58)$$

The non-diagonal elements are a bit trickier to obtain; finding them requires solving the differential equations (20) and (18), that lead to the following system:

$$\begin{aligned} \Omega_{12,x} &= -2\phi_2 \bar{\phi}_1 \\ \Omega_{21,x} &= -2\bar{\phi}_2 \phi_1 \\ \Omega_{12,t} &= -8\bar{\mu} \phi_2 \bar{\phi}_1 + 2\bar{u} \phi_2^2 - 2u \bar{\phi}_1^2 \\ \Omega_{21,t} &= -8\mu \phi_1 \bar{\phi}_2 + 2u \bar{\phi}_2^2 - 2\bar{u} \phi_1^2, \end{aligned} \quad (59)$$

which can be integrated for any particular ϕ_1 and ϕ_2 . In particular, evaluating the matrix Ω for the plane wave (38) with the functions ϕ_i defined as in (41) and (43) one get:

$$\begin{aligned} \Omega_{11} &= -\frac{\eta^2}{A} + \frac{\eta}{A^2} - 4At^2 - \frac{1}{2A^3}, \\ \Omega_{12} &= c_{12} + i \left(\frac{2\eta}{3} - \frac{1}{A} \right) \eta^2 - \frac{16A^3 t^3}{3} - 4iA(2A\eta - 1)t^2 + 2 \left(2A\eta^2 - 2\eta - \frac{1}{A} \right) t, \\ \Omega_{21} &= \bar{\Omega}_{12}, \quad \Omega_{22} = -\Omega_{11}. \end{aligned} \quad (60)$$

and performing the binary DT, described in Section 2, will produce the $n = 2$ multi-rogue wave solutions of [18]. However, the plane wave is just one of possible seed solutions; what would happen should we choose, for example, a *zero* solution $u = 0$?.. Would this produce the same already known multi-soliton rogue waves?.. As we shall see, the answer is no; we will instead get something rather unexpected: a rogue wave that arises during the impact of two positon solitons!

6 The Positon-produced Rogue Wave

In this Section we'll take a particular look at what happens when we use our approach on the one-soliton background instead of a plane wave. As we shall see, in this case the binary DT produces a new rogue wave-like solutions of the NLS, whose scattering profile consists of two “positon” waves.

The first step is traditional: to construct a one soliton solution we start at a zero background $u = 0$. In this case the LA-pairs equations have an extremely simple form

$$\phi_{1,x} = -i\mu\phi_1, \quad \phi_{2,x} = -i\bar{\mu}\phi_2, \quad \phi_{1,t} = -2i\mu^2\phi_1, \quad \phi_{2,t} = -2i(\bar{\mu})^2\phi_2. \quad (61)$$

Solving (61) yields

$$\phi_1 = C_1 e^{-i(\alpha x + 2(\alpha^2 - \beta^2)t) + \beta\xi}, \quad \phi_2 = C_2 e^{-i(\alpha x + 2(\alpha^2 - \beta^2)t) - \beta\xi}, \quad (62)$$

with

$$\mu = \alpha + i\beta, \quad \bar{\mu} = \alpha - i\beta, \quad \xi = x + 4\alpha t, \quad C_1, C_2 \in \mathbb{R}.$$

So, after one DT we end up with the following one soliton solution:

$$u^{(+1)} = -\frac{2i\beta C_1 C_2}{|C_1||C_2|} \frac{e^{-2i(\alpha x + 2(\alpha^2 - \beta^2)t)}}{\cosh\left(2\beta(x + 4\alpha t) + \log\frac{|C_1|}{|C_2|}\right)}. \quad (63)$$

As we have discussed in Section 2, in order to dress (63) via the binary DT one should start by finding the matrix Ω_{ik} . The calculations results in

$$\Omega_{12} = c - 2x - 8\bar{\mu}t = \bar{\Omega}_{21}, \quad \Omega_{22} = -\Omega_{11} = \frac{1}{\beta} \cosh(2\beta\xi), \quad (64)$$

So

$$u^{(+1,-1)} = \frac{4e^{-2i\theta} \left(8\beta t \cosh(2\beta\xi) + i \left(2\xi \sinh(2\beta\xi) - \frac{\cosh(2\beta\xi)}{\beta} \right) \right)}{\left(\frac{1}{\beta} \cosh(2\beta\xi) \right)^2 + 4(\xi^2 + 16\beta^2 t^2)}, \quad (65)$$

with $\theta = \alpha x + 2(\alpha^2 - \beta^2)t$. It is immediately obvious from (65) that this function is even with respect to the variable ξ :

$$u^{(+1,-1)}(-\xi, t) = u^{(+1,-1)}(\xi, t); \quad (66)$$

the importance of this seemingly innocent fact will become apparent a bit later on.

It would be beneficial now to switch from (ξ, t) to a new coordinate system (x, t) , that moves with a velocity $v = -4\alpha$ relative to the old one. In this new system the absolute value of $u^{(+1,-1)}$ turns into

$$|u^{(+1,-1)}|^2 = 16 \frac{64\beta^2 t^2 \cosh^2(2\beta x) + \left(2x \sinh(2\beta x) - \frac{1}{\beta} \cosh(2\beta x) \right)^2}{\left(\frac{1}{\beta} \cosh(2\beta x) \right)^2 + 4(x^2 + 16\beta^2 t^2)}, \quad (67)$$

which is a function that is even with respect to not just one but *both* variables!

$$|u^{(+1,-1)}(x, t)|^2 = |u^{(+1,-1)}(x, t)|^2 = |u^{(+1,-1)}(-x, -t)|^2 = |u^{(+1,-1)}(x, -t)|^2. \quad (68)$$

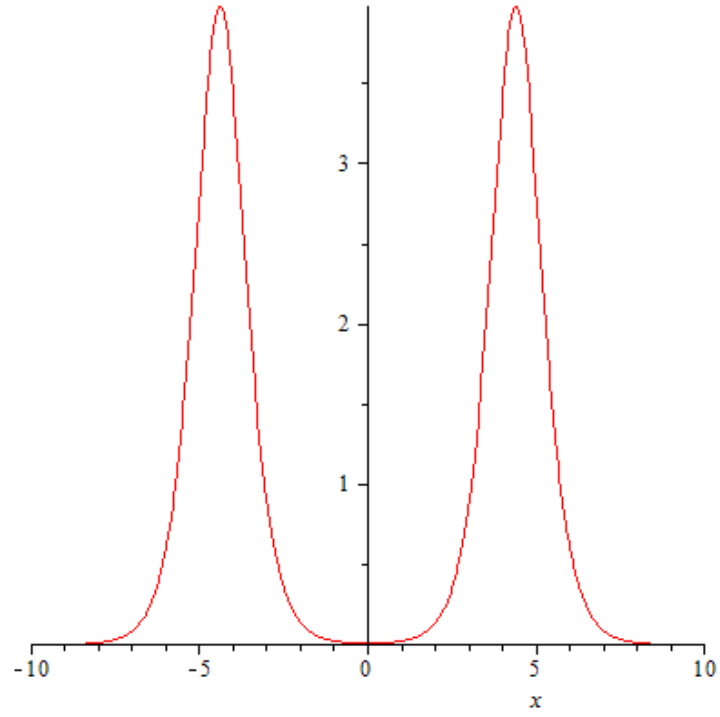


Figure 1: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = -5$. The parameter $\beta = 1$.

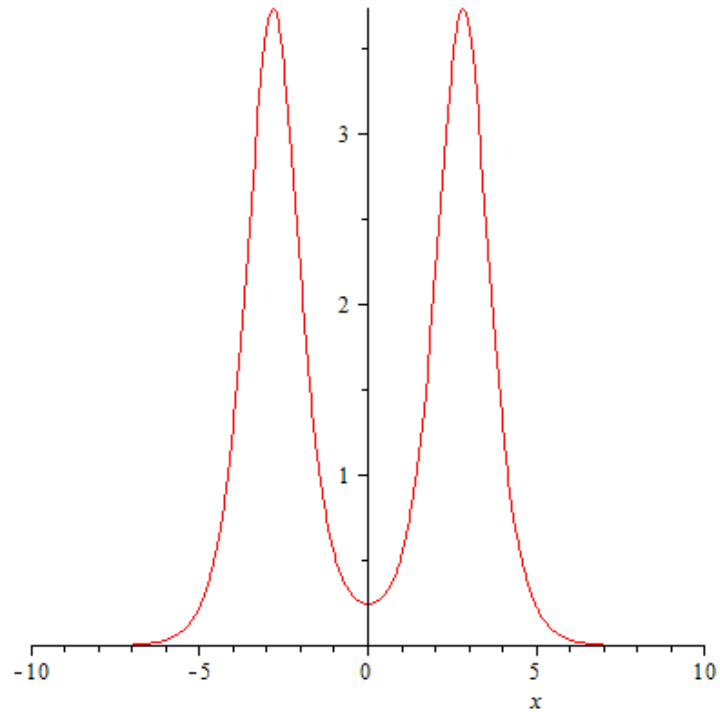


Figure 2: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = -1$.

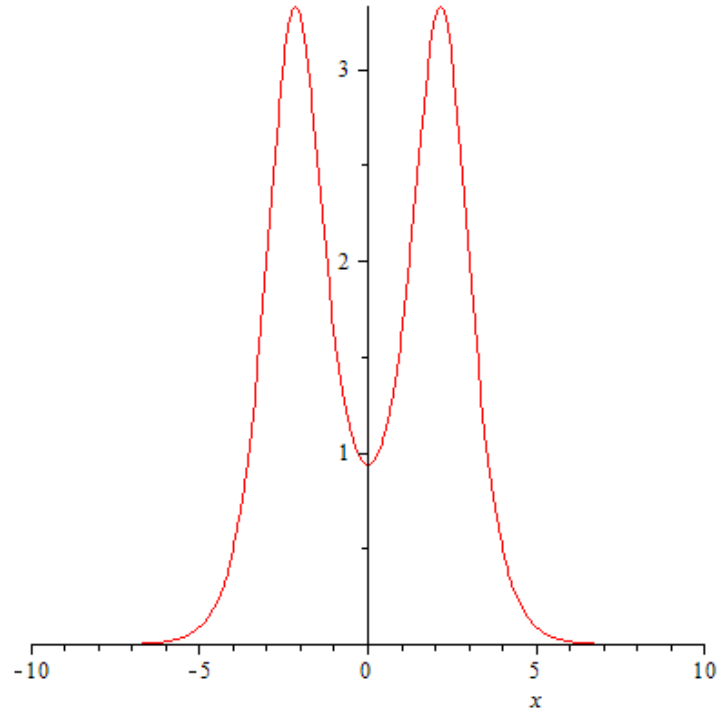


Figure 3: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = -0.5$.

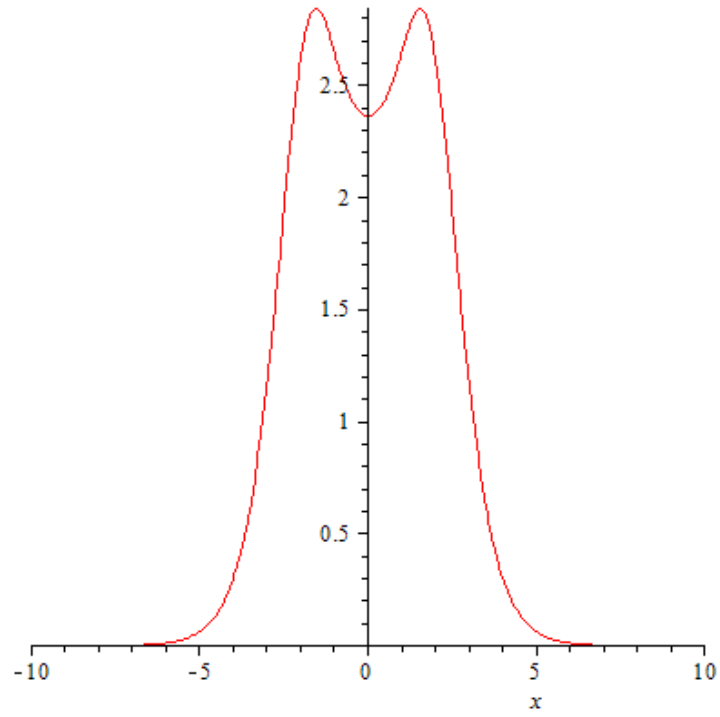


Figure 4: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = -0.3$.

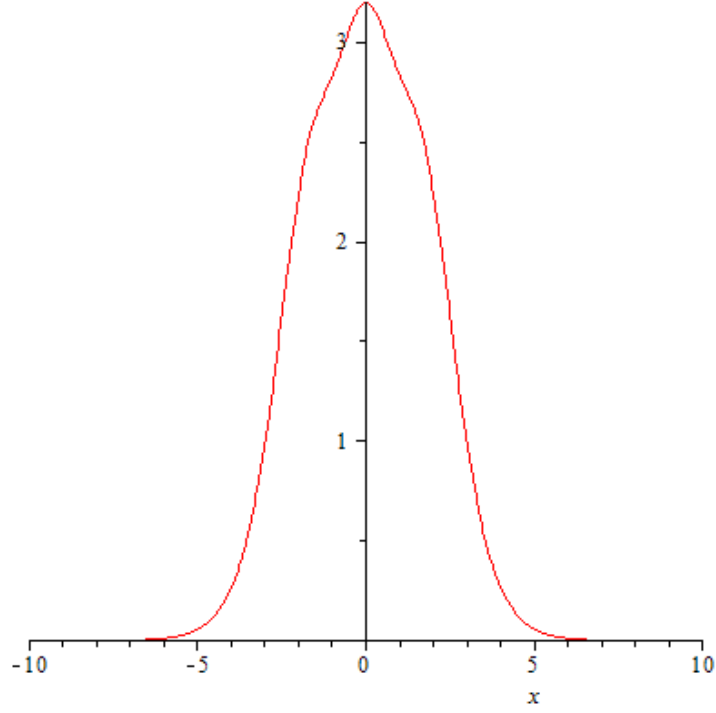


Figure 5: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = -0.25$.

The dynamics of the solution (67) is this: for large absolute values of the variable t it describes two symmetric “solitons” that are slowly approaching each other (see Fig 1).

They impact each other at $t = 0$ and $x = 0$, and it happens in a very unusual fashion: the amplitudes of the solitons crashing into each other begin a rapid decrease (Figs 2-4), at the very point of impact ($x = 0$ and its immediate neighbourhood) a new ephemeral peak emerges (Fig 5).

The amplitude of the peak changes as

$$|u^{(+1,-1)}(0, t)|^2 \sim \frac{16\beta^2}{1 + 64\beta^2 t^2},$$

reaching its maximal value at $t = 0$ (Figs 6-8).

The symmetry of the solution with respect to t results in the process repeating itself for $t > 0$, albeit in reverse: the high slender peak at $x = 0$ is retracted, the individual solitons reemerge and continue moving in their original directions.

Let us now take a closer look at those soliton-like solutions. Let $X = 2\beta x$ and let's assume that at some time $t = t_0 < 0$ the rightmost “soliton” has its maximum localized at some $X = X_0$. The symmetry of the solution implies that the left “soliton” has its maximum located at $X = -X_0$. This means that the derivative of (67) with respect to X should change sign at $X = \pm X_0$, $t = t_0$. This allows us to derive t_0 from X_0 :

$$t_0^2 = \frac{c_0^3 s_0 + 3X_0 c_0^2 - 2X_0^2 s_0 c_0 \pm \sqrt{g_0}}{128\beta^4 s_0 c_0}, \quad (69)$$

where $c_0 = \cosh(X_0)$, $s_0 = \sinh(X_0)$ and

$$g_0 = -8s_0 c_0 X_0^3 - c_0^2 (15c_0^2 - 24) X_0^2 + 6c_0^3 s_0 (4 - c_0^2) X_0 + s_0^2 c_0^4 (c_0^2 + 8).$$

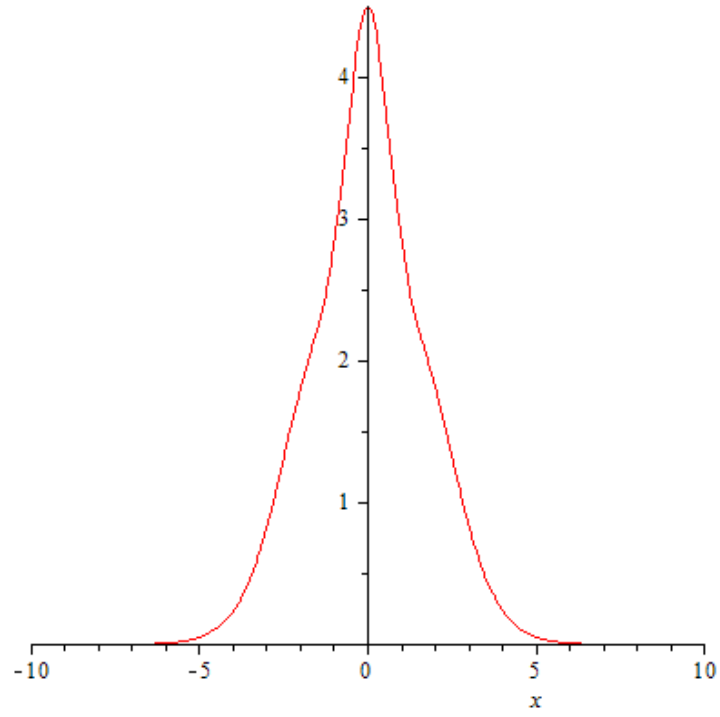


Figure 6: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = -0.2$.

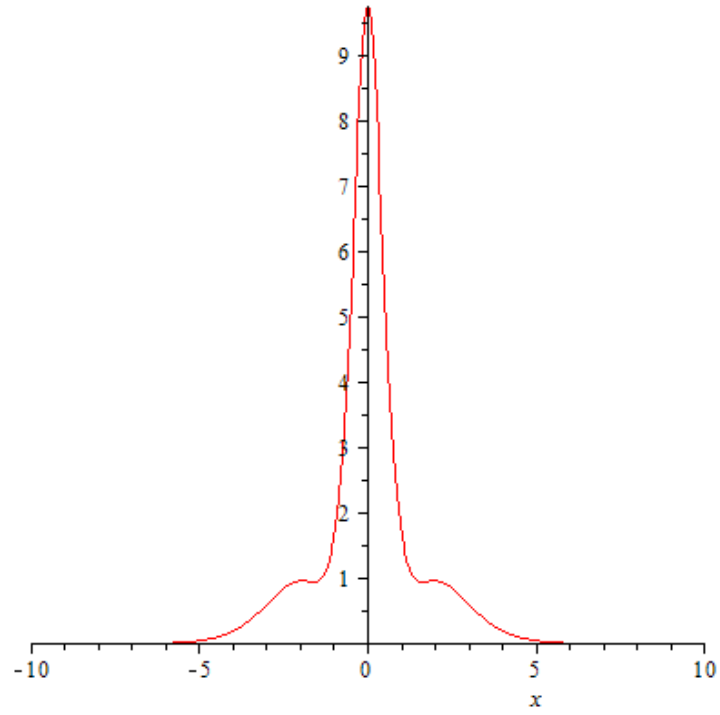


Figure 7: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = -0.1$.

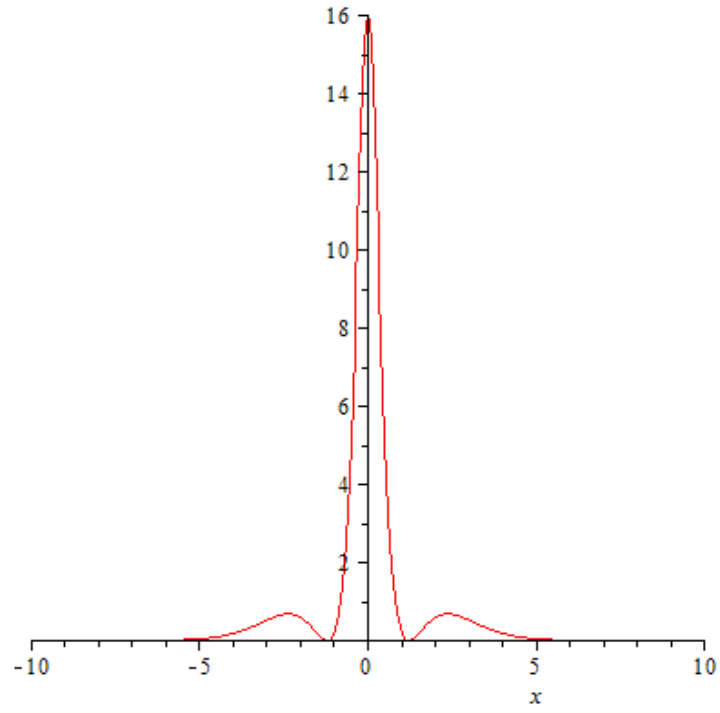


Figure 8: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = 0$.

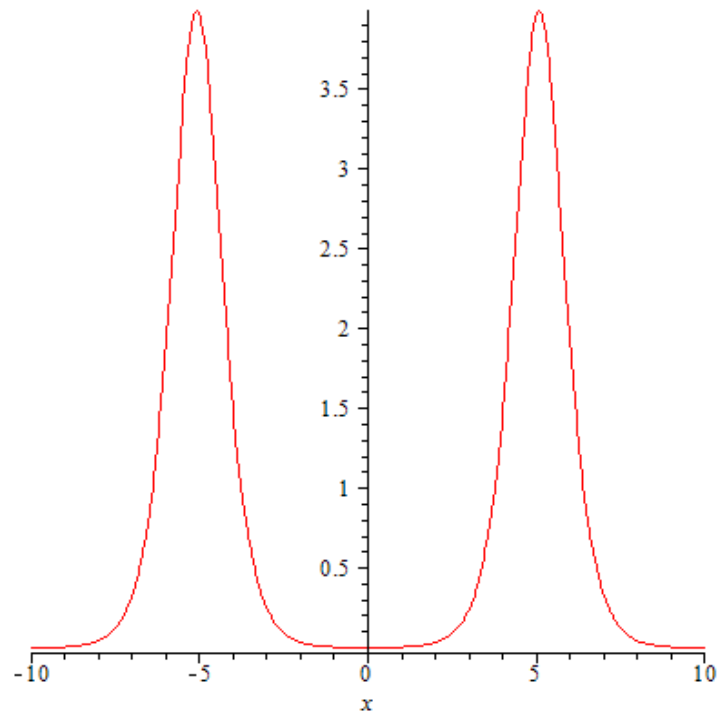


Figure 9: A graph of function $|u^{(+1,-1)}(x, t)|^2$ at $t = +10$.

For sufficiently large X (or t) (69) can be simplified to:

$$t_0 = \frac{c_0}{8\beta^2}. \quad (70)$$

If we substitute this value into (67), we will end up with the following conclusion: for all sufficiently large t the solution consists of two humps that with a very high accuracy (for $X_0 = 10$ the numerical analysis shows the error margin about $10^{-5} - 10^{-7}$) have the form

$$|u|^2 \sim \frac{4\beta^2}{\cosh^2(X - X_0)}. \quad (71)$$

In other words, we are indeed looking at a wave whose dynamics describes two moving soliton solutions of NLS (proportional to a square of a hyperbolic secant) and their collision, which produces a short-lived rogue wave (at the time of impact $t = 0$), whose maximal amplitude is more than twice higher than the amplitudes of the individual solitons (and thus exceeds even their combined height!). The peculiar nature of this rogue wave and the unusual circumstances of its inception warrants it a special name: an “impacton”, denoting an ephemeral rogue wave, born at the point of impact of two colliding soliton waves.

But the surprises do not end here. In fact, the aforementioned individual solitons are quite remarkable even when taken by themselves. As we have seen, outside of the fact their impact begets a rogue wave, when they reemerge after the collision they again behave as two seemingly ordinary NLS solitons, keeping their shape and speed. What is unusual and keeps them apart from other known NLS solitons is the fact that their interaction produces absolutely *no phase shift*.

In order to show this, let's once again look at the Fig. 1. On the picture, we have two NLS solitons moving towards each other. Due to symmetry (66), our solution is invariant under the inversion $x \rightarrow -x$. This implies that both solitons must have identical phase shifts (in fact, these solitons are identical in every respect, with a notable exception of the direction of movement). On the other hand, according to (68), our solution for $t > 0$ is the same as the time-inverted solution at $t < 0$, which means that the collision does not produce any additional phase shifts in either soliton. Summing it all up, we have to conclude that the aforementioned interaction leads to no phase shift.

This unusual property is characteristic of a special breed of solitons, discovered by V. Matveev and called him the “positons” (or “neagtons” – for explanation of the difference between the two see [19]). These solutions have been extensively studied for the KdV equation [20], [17] (but has also been discovered for other types of equations, such as Sine-Gordon and Harry Dym equations [21]–[23]). In particular, it has been shown that the KdV positons correspond to a reflectionless potential and are always singular, which once again makes our solutions stand out as a rather different kettle of fish. We should also point out that also figuring out the exact nature of the solitons (namely, whether they belong to a class of positon or negaton solutions of NLS) in the impacton model lies out of scope of this article, it would definitely be addressed in our subsequent publications.

Finally, what can we say about this solution from a point of view of ferromagnetic spin waves inside of a nanowire?.. As we have discussed in Section 1, the individual x and y components of normalized magnetization $\vec{m} = \vec{M}/M_s$ are just the real and imaginary values of u . However, we have to keep in mind the condition we have imposed on \vec{m} while deriving the NLS equation (6): the function $|u|^2 \ll 1$; in order to achieve this, one should simply choose constant β (see (62)); as we mentioned, the solution (65) reaches its

maximal value at $(x, t) = (0, 0)$. According to (6), this value is equal to $16\beta^2$. This implies that the uppermost value β might theoretically reach is $\beta = 1/4$. However, since we want $\max |u|^2 \ll 1$, the value should be chosen to be even lower; for example, if we choose β to be just ten times smaller, this would mean $\max |u(x, t)|^2 \leq 10^{-2}$, which should satisfy our requirements just fine.

In conclusion, we would like to point out that the method proposed in this article allows for a number of generalizations; in particular, it can be utilized for the case of a one-dimensional multicomponent magnonic crystal with each the dynamics of each component described by the NLS (10) with its own coefficients. Using the binary Darboux on the background of planar wave for each of those components would produce the solutions with a number of free parameters that can be defined to “stitch” together the solutions from the different components, thus describing the movement of a P-breather through the entire crystal. The exact techniques for this case will be described in our next work.

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